

A finite-sum representation for solutions for the Jacobi operator

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Abstract

We obtain a finite-sum representation for the general solution of the equation

$$\Delta(p(n-1)\Delta u(n-1)) + q(n)u(n) = \lambda r(n)u(n)$$

in terms of a nonvanishing solution corresponding to some fixed value of $\lambda = \lambda_0$. Applications of this representation to some results on the boundedness of solutions are given as well as illustrating examples.

Keywords: Jacobi operator; difference equation; spectral parameter power series

1 Introduction

First we introduce some notations. For $a \in \mathbb{R}$ we define the following set $N_a = \{a, a+1, \dots\}$, and Δ stays for the difference operator, $\Delta u(n) = u(n+1) - u(n)$. A sequence $v(n)$ satisfying $\Delta v(n) = u(n)$ is called an indefinite sum of $u(n)$. The indefinite sum of a sequence is not unique and corresponding indefinite sums differ by a constant. By $\sum_{j=n_0}^{n-1} *u(j)$ we denote the indefinite sum of

$u(n)$ satisfying the boundary condition $u(n_0) = 0$. Then we have [11]:

$$\sum_{j=n_0}^{n-1} {}^*u(j) = \begin{cases} \sum_{j=n_0}^{n-1} u(j), & n > n_0 \\ 0, & n = n_0 \\ -\sum_{j=n}^{n_0-1} u(j), & n < n_0 \end{cases}.$$

We will consider the second order difference equation of the form

$$\Delta(p(n-1)\Delta u(n-1)) + q(n)u(n) = \lambda r(n)u(n), n \in N_a, \quad (1)$$

where r, p, q are given complex sequences defined on N_a, N_{a-1} and N_a respectively, $p(n) \neq 0$ for all $n \in N_{a-1}$, $\lambda \in \mathbb{C}$ is the spectral parameter and $u(n)$, defined on N_{a-1} , is the unknown function. The operator on the left-hand side is known as the Jacobi operator and has been extensively studied (see, e.g., [11]). If in (1) $\lambda = 0$, we obtain the equation

$$\Delta(p(n-1)\Delta u(n-1)) + q(n)u(n) = 0 \quad (2)$$

which was studied in dozens of works regarding several aspects, namely, oscillation, disconjugacy, disfocality, asymptotic behaviour, boundedness and boundary value problem (see, e.g., the books [1], [2], [6] and [7]). Equation (1) can be regarded as a discrete analogue of the Sturm-Liouville differential equation

$$(p(x)y'(x))' + q(x)y(x) = \lambda r(x)y(x)$$

and quite often techniques and results developed for (1) represent discrete analogues of the corresponding continuous results for the Sturm-Liouville equation.

In this paper we begin by obtaining a discrete version of some results from [9] and [10] concerning the spectral power series representation for the general solution of the Sturm-Liouville differential equation. This representation being a different form of a perturbation Liouville-Neumann series [3] offers an efficient algorithm for numerical calculation of eigenfunctions and eigenvalues of a Sturm-Liouville problem (see [4], [9], [10], [8]).

For linear difference equations a spectral power series representation for solutions was considered also as a perturbation technique, however even the situation with the convergence of such series was not satisfactorily understood (see, e.g., [1, p. 91], where the possibility of divergence of the series as those considered in the present work is assumed). Motivated by [10] and [9], we propose a different procedure to find the coefficients of such series for

solutions of (1) (see Theorem 1) which gives us as a simple corollary that those series are in fact finite sums (see Lemma 1).

As an application of this representation, we give alternative proofs of some results already known in the literature, concerning boundedness of solutions. We also extend the criterion of the boundedness of all solutions of a linear second-order difference equation onto a general case of complex coefficients, Theorem 2.

2 A finite-sum representation for solutions

In this section we prove the main result of the present work, Theorem 1, which establishes that any nonvanishing solution u_0 of (2) allows us to obtain a general solution of (1) as follows. Consider the sequences

$$u_1(n) = \begin{cases} u_0(n) \sum_{k=0}^{n-n_0-1} \lambda^k X^{(2k)}(n), & n > n_0 \\ u_0(n) \sum_{k=0}^{n_0-n} \lambda^k X^{(2k)}(n), & n \leq n_0 \end{cases} \quad (3)$$

and

$$u_2(n) = \begin{cases} u_0(n) \sum_{k=0}^{|n-n_0|-1} \lambda^k Y^{(2k+1)}(n), & n \neq n_0 \\ 0, & n = n_0 \end{cases} \quad (4)$$

where $X^{(i)}$ and $Y^{(i)}$ are defined recursively by the relations

$$X^{(0)} = Y^{(0)} = 1,$$

$$X^{(i)}(n) = \begin{cases} \sum_{s=n_0}^{n-1} * \frac{X^{(i-1)}(s)}{p(s)u_0(s)u_0(s+1)}, & i \text{ even} \\ \sum_{s=n_0}^{n-1} * u_0^2(s+1)X^{(i-1)}(s+1)r(s+1), & i \text{ odd} \end{cases} \quad (5)$$

$$Y^{(i)}(n) = \begin{cases} \sum_{s=n_0}^{n-1} * u_0^2(s+1)Y^{(i-1)}(s+1)r(s+1), & i \text{ even} \\ \sum_{s=n_0}^{n-1} * \frac{Y^{(i-1)}(s)}{p(s)u_0(s)u_0(s+1)}, & i \text{ odd} \end{cases} \quad (6)$$

and $n_0 \in N_{a-1}$ is an arbitrary point. We show that they are linearly independent solutions of (1). In order to prove this statement we need first the following auxiliary result.

Lemma 1.

(i) For $k \geq 1$,

$$m \in \{n_0 - k + 1, \dots, n_0 + k\} \Rightarrow X^{(2k)}(m) = 0,$$

and for $k \geq 0$,

$$m \in \{n_0, n_0 \pm 1, \dots, n_0 \pm k\} \Rightarrow Y^{(2k+1)}(m) = 0.$$

(ii) The sequences defined by the formulas

$$u_1(n) = u_0(n) \sum_{k=0}^{\infty} \lambda^k X^{(2k)}(n), u_2(n) = u_0(n) \sum_{k=0}^{\infty} \lambda^k Y^{(2k+1)}(n)$$

can be written as follows

$$u_1(n) = \begin{cases} u_0(n) \sum_{k=0}^{n-n_0-1} \lambda^k X^{(2k)}(n), & n > n_0 \\ u_0(n) \sum_{k=0}^{n_0-n} \lambda^k X^{(2k)}(n), & n \leq n_0 \end{cases}$$

$$u_2(n) = \begin{cases} u_0(n) \sum_{k=0}^{|n-n_0|-1} \lambda^k Y^{(2k+1)}(n), & n \neq n_0 \\ 0, & n = n_0 \end{cases}.$$

Proof. (i) We use the reasoning by induction to prove that if $m \in \{n_0 + 1, \dots, n_0 + k\}$ then

$$X^{(2k)}(m) = 0, \quad k > 0. \quad (7)$$

All other cases contemplated in (i) are treated in a similar way. Note that by definition $X^{(i)}(n_0) = 0$ for all $i \neq 0$. For $k = 1$ relation (7) holds due to the equality

$$X^{(2)}(n_0 + 1) = \frac{X^{(1)}(n_0)}{p(n_0)u_0(n_0)u_0(n_0 + 1)} = 0.$$

Suppose that the assertion is true for k . Then by the equality

$$X^{(2k+1)}(n) = \sum_{s=n_0}^{n-1} u_0^2(s+1)X^{(2k)}(s+1)r(s+1), \quad n > n_0,$$

we conclude that $n_0 + 1, \dots, n_0 + k$ are zeros of $X^{(2k+1)}$. From this and due to the relation

$$X^{(2k+2)}(n) = \sum_{s=n_0}^{n-1} \frac{X^{(2k+1)}(s)}{p(s)u_0(s)u_0(s+1)}, \quad n > n_0$$

we obtain that $n_0 + 1, \dots, n_0 + k + 1$ are zeros of $X^{(2k+2)}$, and the assertion is valid for $k + 1$. (ii) Let $n > n_0$. By part (i), $X^{(2k)}(n) = 0$ for all $k \geq n - n_0$, thus the series defining u_1 is actually a sum from $k = 0$ to $k = n - n_0 - 1$. Other cases are proved similarly. \square

Theorem 1. Assume that u_0 is a nonvanishing solution of (2). Then the sequences (3) and (4) are linearly independent solutions of (1), where $X^{(i)}$ and $Y^{(i)}$ are defined recursively by the relations (5) and (6) and $n_0 \in N_{a-1}$ is an arbitrary point.

Proof. First we prove that the sequences u_1 and u_2 defined as follows

$$u_1(n) = u_0(n) \sum_{k=0}^{\infty} \lambda^k X^{(2k)}(n), u_2(n) = u_0(n) \sum_{k=0}^{\infty} \lambda^k Y^{(2k+1)}(n) \quad (8)$$

satisfy equation (1) and are linearly independent. As was shown in Lemma 1 these infinite series are in fact the finite sums (3), (4). With the help of the nonvanishing solution $u_0(n)$ of (2) the Jacobi operator

$$Lu(n) = \Delta(p(n-1)\Delta u(n-1)) + q(n)u(n)$$

can be factorized as follows

$$Lu(n) = \frac{1}{u_0(n)} \Delta \left[p(n-1)u_0(n-1)u_0(n) \Delta \left(\frac{u(n-1)}{u_0(n-1)} \right) \right]$$

(this is the Polya factorization [2], [7]). Applying the operator L to u_1 we obtain

$$\begin{aligned} Lu_1(n) &= \frac{1}{u_0(n)} \Delta \left[p(n-1)u_0(n-1)u_0(n) \Delta \sum_{k=0}^{\infty} \lambda^k X^{(2k)}(n-1) \right] \\ &= \frac{1}{u_0(n)} \Delta \sum_{k=1}^{\infty} \lambda^k X^{(2k-1)}(n-1) \\ &= r(n)u_0(n) \sum_{k=1}^{\infty} \lambda^k X^{(2k-2)}(n) = \lambda r(n)u_1(n). \end{aligned}$$

The same technique can be used to prove that $u_2(n)$ is a solution as well. In order to prove that u_1 and u_2 are linearly independent, it is necessary to verify that their Casoratian is different from zero at any point. Since $u_2(n_0) = 0$, the Casoratian of u_1 and u_2 at n_0 can be calculated,

$$W(u_1, u_2)(n_0) = u_1(n_0)u_2(n_0 + 1) = \frac{u_0(n_0)}{p(n)u_0(n_0)} = \frac{1}{p(n_0)} \neq 0.$$

□

Example 1. Consider the equation $\Delta^2 u(n-1) = \lambda u(n)$. In this case one can choose $u_0 \equiv 1$ and $n_0 = 0$. Then by (5) and (6) we have

$$X^{(2k)}(n) = \frac{(n+k-1)^{(2k)}}{2k!}, \quad Y^{(2k+1)}(n) = \frac{(n+k)^{(2k+1)}}{(2k+1)!},$$

where $n^{(k)} := n(n-1)\dots(n-k+1)$.

Remark 1. Obviously Theorem 1 can also be applied when a nonvanishing solution of the equation

$$\Delta(p(n-1)\Delta u(n-1)) + q(n)u(n) = \lambda_0 r(n)u(n)$$

is known. In this case the Polya factorization is applied to the operator $\tilde{T} = T - \lambda_0 I$ and (1) is written in the form $\tilde{T}u = (\lambda - \lambda_0)u$. Then for $n > n_0$ the solutions (3) and (4) are given by the following sums

$$u_1(n) = u_0(n) \sum_{k=0}^{n-n_0-1} (\lambda - \lambda_0)^k X^{(2k)}(n), \quad u_2(n) = u_0(n) \sum_{k=0}^{n-n_0-1} (\lambda - \lambda_0)^k Y^{(2k+1)}(n),$$

and for $n \leq n_0$ the corresponding representation of solutions is also obtained from (3) and (4) by replacing λ^k with $(\lambda - \lambda_0)^k$.

Remark 2. When p and q are real sequences a nonvanishing solution always exists. Indeed, two linearly independent real solutions u and v never vanish simultaneously (because otherwise their Casoratian vanishes) thus one can choose $u_0 = u + iv$.

Example 2. Let us consider the equation

$$\Delta(n\Delta u(n-1)) + \lambda u(n) = 0, \quad n \in N_1. \quad (9)$$

Let $n_0 = 0$ and $u_0 \equiv 1$. Using the auxiliary operator

$$Tu(n) := \sum_{s=0}^{n-1} * \frac{1}{1+s} \sum_{l=0}^{s-1} * u(l+1),$$

from (5) and (6) we get $X^{(2k)} = T(X^{(2k-2)})$ and $X^{(2k+1)} = T(X^{(2k-1)})$. Then we have the following relations

$$\begin{aligned} X^{(2)}(n) &= T(1) = \sum_{s=0}^{n-1} \frac{s}{1+s} = \sum_{s=0}^{n-1} 1 - \sum_{s=0}^{n-1} \frac{1}{1+s} = n - Y^{(1)}(n), \\ X^{(4)}(n) &= T(n) - T(Y^{(1)}(n)) = \frac{n^{(2)}}{4} - Y^{(3)}(n), \\ &\vdots \\ X^{(2k)}(n) &= \frac{n^{(k)}}{(k!)^2} - Y^{(2k-1)}(n). \end{aligned} \quad (10)$$

Consider the following combination of the two solutions

$$u(n, \lambda) := u_1(n) - \lambda u_2(n) = (-\lambda)^n Y^{(2n-1)}(n) + \sum_{k=0}^{n-1} (-\lambda)^k \frac{n^{(k)}}{(k!)^2}.$$

By Lemma 1 $X^{(2n)}(n) = 0$, and due to (10) we have $Y^{(2n-1)}(n) = \frac{n^{(n)}}{(n!)^2}$. Thus,

$$u(n, \lambda) = \sum_{k=0}^n (-\lambda)^k \frac{n^{(k)}}{(k!)^2} = \sum_{k=0}^n \binom{n}{k} \frac{(-\lambda)^k}{k!}, \quad n \geq 0.$$

Note that these are the Laguerre polynomials (of the variable λ) and as is well known they satisfy (9) (see, e.g., [6]).

Let us distinguish the following special case of Theorem 1.

Corollary 1. *The sequences u_1 , u_2 defined by (3), (4), (5) and (6) with $u_0 \equiv \lambda = 1$ are linearly independent solutions of the equation*

$$\Delta(p(n-1)\Delta u(n-1)) = r(n)u(n), \quad n \in N_a. \quad (11)$$

Remark 3. *Let u_1 and u_2 be the solutions from the above corollary. Then for $n > n_0$ we have*

$$u_1(n) = \sum_{k=0}^{n-n_0-1} X^{(2k)}(n), \quad u_2(n) = \sum_{k=0}^{n-n_0-1} Y^{(2k+1)}(n). \quad (12)$$

Note that using the operator defined by

$$Tu(n) := \sum_{s=n_0+1}^{n-1} \sum_{\tau=n_0+1}^s \frac{u(\tau)r(\tau)}{p(s)}, \quad n > n_0 + 1, \quad (13)$$

we obtain

$$X^{(2k)} = T(X^{(2k-2)}), \quad Y^{(2k+1)} = T(Y^{(2k-1)}). \quad (14)$$

3 Applications to results on the boundedness of solutions

We begin giving another (in our opinion, an easier) proof of an important result obtained in [5].

Proposition 1. *Let $p(n) > 0$ and $r(n) \geq 0$. If all solutions of (11) are bounded then*

$$\sum_{s=a}^{\infty} \sum_{\tau=a}^s \frac{r(\tau)}{p(s)} < \infty, \quad \sum_{s=a}^{\infty} \frac{1}{p(s)} < \infty.$$

Proof. Since the sequences p and q are nonnegative, by definition $X^{(2k)}$ and $Y^{(2k+1)}$ are nonnegative as well (see Remark 3), and as all solutions of (11) are bounded, then of course

$$X^{(2)}(n) = T(1) = \sum_{s=n_0+1}^{n-1} \sum_{\tau=n_0+1}^s \frac{r(\tau)}{p(s)} \quad \text{and} \quad Y^{(1)}(n) = \sum_{s=n_0}^{n-1} \frac{1}{p(s)} \quad (15)$$

are bounded too. \square

The converse of the above result was also proved in [5]. The proof given there works only in the case of nonnegative coefficients. We prove a more general result.

Theorem 2. *Let $p(n) \neq 0$. If*

$$\sum_{s=a}^{\infty} \sum_{\tau=a}^s \frac{|r(\tau)|}{|p(s)|} < \infty, \quad \sum_{s=a}^{\infty} \frac{1}{|p(s)|} < \infty \quad (16)$$

then all solutions of (11) are bounded.

Proof. Assume that the condition (16) is fulfilled. Then by (15) there exists some n_0 such that both $|Y^1(n)|$ and $|X^{(2)}(n)|$ are less than $\delta < 1$ for any $n > n_0$. By (13), for $n > n_0 + 1$ we have

$$|Tu(n)| < \sup |X^{(2)}(n)| \cdot \sup |u(n)|.$$

From this and by (14) we get $|Y^{(2k-1)}(x)|, |X^{(2k)}(n)| < \delta^k$. Then from (12) we obtain the boundedness of solutions. \square

It is known [5] that minimal solutions of (11) under the conditions of Proposition 1 and when the condition (16) is not fulfilled, tend to zero iff there exists such a solution u of (11) that the sequence $p(n)\Delta u(n)$ is unbounded. The existence of such solutions is completely described by the following proposition to which we also give another and easier proof.

Proposition 2. *Let $p(n) > 0$ and $r(n) \geq 0$. Then for every solution $u(n)$ of (11) the function $\phi(n) = p(n)\Delta u(n)$ is bounded if and only if*

$$\sum_{s=a}^{\infty} \sum_{\tau=a}^s \frac{r(s+1)}{p(\tau)} < \infty. \quad (17)$$

Proof. Denote $\phi_{1,2}(n) = p(n)\Delta u_{1,2}(n)$, where $u_{1,2}$ are given by (12). Then

$$\phi_1(n) = \sum_{k=0}^{n-n_0-1} X^{(2k+1)}(n), \quad \phi_2(n) = \sum_{k=0}^{n-n_0} Y^{(2k)}(n).$$

Furthermore, the relations $X^{(2k+1)} = \tilde{T}(X^{(2k-1)})$ and $Y^{(2k+2)} = \tilde{T}(Y^{(2k)})$ hold, where \tilde{T} is the operator defined by

$$\tilde{T}u(n) := \sum_{s=n_0}^{n-1} \sum_{\tau=n_0}^s \frac{r(s+1)u(\tau)}{p(\tau)}.$$

Then the sufficiency of condition (17) is proved following the reasoning from the proof of Proposition 1, and the necessity is proved following the reasoning from the proof of Theorem 2. \square

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